

Box 2-2 Subtleties

We first need to recall the definition of physical 3-momentum

$$\begin{aligned} p^i &= m\dot{x}^i = m \frac{dx^i}{dt} & \left\{ \begin{array}{l} \text{contravariant form is physical 3-momentum we measure} \\ \text{(found from } x^i, \text{ the physical coords we measure)} \end{array} \right. \\ p_i &= m\dot{x}_i = m \frac{dx_i}{dt} = -m \frac{dx^i}{dt} = -m\dot{x}^i & \left\{ \begin{array}{l} \text{covariant form is negative of physical 3-momentum} \\ \text{(is not what we measure for 3-momentum)} \end{array} \right. \end{aligned} \quad (1)$$

Then consider our free total Lagrangian L (no potential energy)

$$L = \frac{1}{2} m (\dot{x}^i)^2, \quad (2)$$

and its conjugate momentum (as usually defined in tensor analysis so the subscript i before the first equal sign is equivalent to a superscript i in the denominator after the equal sign),

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \quad \left\{ \begin{array}{l} \text{conjugate momentum via tensor analysis.} \end{array} \right. \quad (3)$$

Note that the p_i in (3) differs from that in the second row of (1)! In reality, the value of the conjugate momentum equals our physical 3-momentum (top row of (1)).

So what we show in the LHS of (B2-2.1) in Box 2-2 is the conjugate momentum p_i , but its actual value $m\dot{x}^i$ is what we use the symbol p^i for in the rest of the book, i.e., the real physical 3-momentum (not its negative).

So to stay consistent with the symbolism of the rest of the book (and in the literature) we really should have written (B-2-2.1) as

$$p_i = -\frac{\partial L}{\partial \dot{x}^i} \xrightarrow[\text{divide by particle volume}]{\text{for small particle in medium,}} \mathcal{P}_i = -\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \quad \left\{ \begin{array}{l} \text{(B2-2.1) re-written in terms of} \\ \text{symbols used elsewhere in book,} \\ \text{opposite sign of tensor analysis} \end{array} \right. \quad (4)$$

(This would have been inconsistent with the usual tensor analysis notation.)

Similar logic holds for the conjugate momentum π_r found from the Lagrangian density \mathcal{L} in (B2-2.2). That is, its appropriate value corresponding to the symbolism of (4) has the opposite sign of what Box 2-2 shows, i.e.,

$$\pi_r = -\frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} \quad \left\{ \begin{array}{l} \text{(B2-2.2) re-written,} \\ \text{opposite sign of tensor analysis} \end{array} \right. \quad (5)$$

If we use the symbolism of (4) and (5), which parallels the rest of the book and other literature in QFT, then in (B2-2.3), reproduced below, the minus signs drop out and we get the same result.

$$\frac{\mathcal{P}_i}{\pi_r} = \frac{\partial \mathcal{L} / \partial \dot{x}^i}{\partial \mathcal{L} / \partial \dot{\phi}^r} = \frac{\partial \dot{\phi}^r}{\partial \dot{x}^i} = \frac{\partial \phi^r / \partial t}{\partial x^i / \partial t} = \frac{\partial \phi^r}{\partial x^i} \rightarrow \mathcal{P}_i = \pi_r \frac{\partial \phi^r}{\partial x^i} \rightarrow \mathcal{P}^i = -\pi_r \frac{\partial \phi^r}{\partial x^i} \quad \left\{ \begin{array}{l} \text{(B2-2.3),} \\ \text{same result} \end{array} \right. \quad (6)$$

Perhaps now you can see why I fudged a bit in the book Box 2-2, and why I didn't want to get into this explanation there, at that time. It would have confused newcomers far more than enlightened them.